

### 3.3 REAL ZEROS OF POLYNOMIALS

In Section 3.2, we found that we can use synthetic division to determine if a given real number is a zero of a polynomial function. This section presents results which will help us determine good candidates to test using synthetic division. There are two approaches to the topic of finding the real zeros of a polynomial. The first approach (which is gaining popularity) is to use a little bit of Mathematics followed by a good use of technology like graphing calculators. The second approach (for purists) makes good use of mathematical machinery (theorems) only. For completeness, we include the two approaches but in separate subsections.<sup>1</sup> Both approaches benefit from the following two theorems, the first of which is due to the famous mathematician Augustin Cauchy. It gives us an interval on which all of the real zeros of a polynomial can be found.

**Theorem 3.8. Cauchy's Bound:** Suppose  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a polynomial of degree  $n$  with  $n \geq 1$ . Let  $M$  be the largest of the numbers:  $\frac{|a_0|}{|a_n|}, \frac{|a_1|}{|a_n|}, \dots, \frac{|a_{n-1}|}{|a_n|}$ . Then all the real zeros of  $f$  lie in the interval  $[-(M + 1), M + 1]$ .

The proof of this fact is not easily explained within the confines of this text. This paper contains the result and gives references to its proof. Like many of the results in this section, Cauchy's Bound is best understood with an example.

**Example 3.3.1.** Let  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ . Determine an interval which contains all of the real zeros of  $f$ .

**Solution.** To find the  $M$  stated in Cauchy's Bound, we take the absolute value of leading coefficient, in this case  $|2| = 2$  and divide it into the largest (in absolute value) of the remaining coefficients, in this case  $|-6| = 6$ . We find  $M = 3$ , so it is guaranteed that the real zeros of  $f$  all lie in  $[-4, 4]$ .  $\square$

Whereas the previous result tells us *where* we can find the real zeros of a polynomial, the next theorem gives us a list of *possible* real zeros.

**Theorem 3.9. Rational Zeros Theorem:** Suppose  $f(x) = a_n x^n + a_{n-1} x^{n-1} + \dots + a_1 x + a_0$  is a polynomial of degree  $n$  with  $n \geq 1$ , and  $a_0, a_1, \dots, a_n$  are integers. If  $r$  is a rational zero of  $f$ , then  $r$  is of the form  $\pm \frac{p}{q}$ , where  $p$  is a factor of the constant term  $a_0$ , and  $q$  is a factor of the leading coefficient  $a_n$ .

The Rational Zeros Theorem gives us a list of numbers to try in our synthetic division and that is a lot nicer than simply guessing. If none of the numbers in the list are zeros, then either the polynomial has no real zeros at all, or all of the real zeros are irrational numbers. To see why the Rational Zeros Theorem works, suppose  $c$  is a zero of  $f$  and  $c = \frac{p}{q}$  in lowest terms. This means  $p$  and  $q$  have no common factors. Since  $f(c) = 0$ , we have

$$a_n \left(\frac{p}{q}\right)^n + a_{n-1} \left(\frac{p}{q}\right)^{n-1} + \dots + a_1 \left(\frac{p}{q}\right) + a_0 = 0.$$

<sup>1</sup>Carl is the purist and is responsible for all of the theorems in this section. Jeff, on the other hand, has spent too much time in school politics and has been polluted with notions of 'compromise.' You can blame the slow decline of civilization on him and those like him who mingle Mathematics with technology.

Multiplying both sides of this equation by  $q^n$ , we clear the denominators to get

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = 0$$

Rearranging this equation, we get

$$a_n p^n = -a_{n-1} p^{n-1} q - \dots - a_1 p q^{n-1} - a_0 q^n$$

Now, the left hand side is an integer multiple of  $p$ , and the right hand side is an integer multiple of  $q$ . (Can you see why?) This means  $a_n p^n$  is both a multiple of  $p$  and a multiple of  $q$ . Since  $p$  and  $q$  have no common factors,  $a_n$  must be a multiple of  $q$ . If we rearrange the equation

$$a_n p^n + a_{n-1} p^{n-1} q + \dots + a_1 p q^{n-1} + a_0 q^n = 0$$

as

$$a_0 q^n = -a_n p^n - a_{n-1} p^{n-1} q - \dots - a_1 p q^{n-1}$$

we can play the same game and conclude  $a_0$  is a multiple of  $p$ , and we have the result.

**Example 3.3.2.** Let  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ . Use the Rational Zeros Theorem to list all of the possible rational zeros of  $f$ .

**Solution.** To generate a complete list of rational zeros, we need to take each of the factors of constant term,  $a_0 = -3$ , and divide them by each of the factors of the leading coefficient  $a_4 = 2$ . The factors of  $-3$  are  $\pm 1$  and  $\pm 3$ . Since the Rational Zeros Theorem tacks on a  $\pm$  anyway, for the moment, we consider only the positive factors 1 and 3. The factors of 2 are 1 and 2, so the Rational Zeros Theorem gives the list  $\{\pm \frac{1}{1}, \pm \frac{1}{2}, \pm \frac{3}{1}, \pm \frac{3}{2}\}$  or  $\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\}$ .  $\square$

Our discussion now diverges between those who wish to use technology and those who do not.

### 3.3.1 FOR THOSE WISHING TO USE A GRAPHING CALCULATOR

At this stage, we know not only the interval in which all of the zeros of  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$  are located, but we also know some potential candidates. We can now use our calculator to help us determine all of the real zeros of  $f$ , as illustrated in the next example.

**Example 3.3.3.** Let  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ .

1. Graph  $y = f(x)$  on the calculator using the interval obtained in Example 3.3.1 as a guide.
2. Use the graph to shorten the list of possible rational zeros obtained in Example 3.3.2.
3. Use synthetic division to find the real zeros of  $f$ , and state their multiplicities.

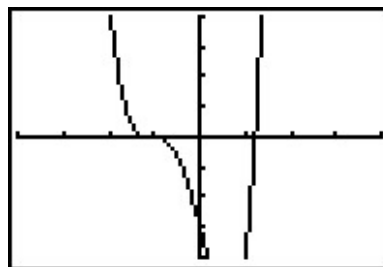
**Solution.**

1. In Example 3.3.1, we determined all of the real zeros of  $f$  lie in the interval  $[-4, 4]$ . We set our window accordingly and get

```

WINDOW
Xmin=-4
Xmax=4
Xscl=1
Ymin=-4
Ymax=4
Yscl=1
Xres=1

```



2. In Example 3.3.2, we learned that any rational zero of  $f$  must be in the list  $\{\pm \frac{1}{2}, \pm 1, \pm \frac{3}{2}, \pm 3\}$ . From the graph, it looks as if we can rule out any of the positive rational zeros, since the graph seems to cross the  $x$ -axis at a value just a little greater than 1. On the negative side,  $-1$  looks good, so we try that for our synthetic division.

$$\begin{array}{r|rrrrr}
 -1 & 2 & 4 & -1 & -6 & -3 \\
 & \downarrow & -2 & -2 & 3 & 3 \\
 \hline
 & 2 & 2 & -3 & -3 & \boxed{0}
 \end{array}$$

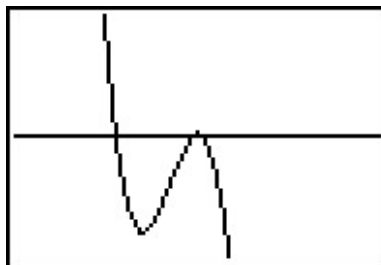
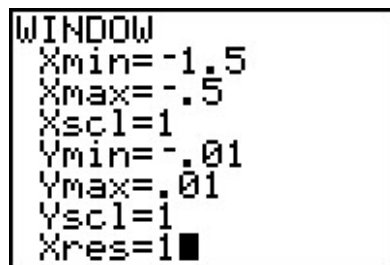
We have a winner! Remembering that  $f$  was a fourth degree polynomial, we know that our quotient is a third degree polynomial. If we can do one more successful division, we will have knocked the quotient down to a quadratic, and, if all else fails, we can use the quadratic formula to find the last two zeros. Since there seems to be no other rational zeros to try, we continue with  $-1$ . Also, the shape of the crossing at  $x = -1$  leads us to wonder if the zero  $x = -1$  has multiplicity 3.

$$\begin{array}{r|rrrrr}
 -1 & 2 & 4 & -1 & -6 & -3 \\
 & \downarrow & -2 & -2 & 3 & 3 \\
 \hline
 -1 & 2 & 2 & -3 & -3 & \boxed{0} \\
 & \downarrow & -2 & 0 & 3 & \\
 \hline
 & 2 & 0 & -3 & \boxed{0}
 \end{array}$$

Success! Our quotient polynomial is now  $2x^2 - 3$ . Setting this to zero gives  $2x^2 - 3 = 0$ , or  $x^2 = \frac{3}{2}$ , which gives us  $x = \pm \frac{\sqrt{6}}{2}$ . Concerning multiplicities, based on our division, we have that  $-1$  has a multiplicity of at least 2. The Factor Theorem tells us our remaining zeros,  $\pm \frac{\sqrt{6}}{2}$ , each have multiplicity at least 1. However, Theorem 3.7 tells us  $f$  can have at most 4 real zeros, counting multiplicity, and so we conclude that  $-1$  is of multiplicity exactly 2 and  $\pm \frac{\sqrt{6}}{2}$  each has multiplicity 1. (Thus, we were wrong to think that  $-1$  had multiplicity 3.)  $\square$

It is interesting to note that we could greatly improve on the graph of  $y = f(x)$  in the previous example given to us by the calculator. For instance, from our determination of the zeros of  $f$  and their multiplicities, we know the graph crosses at  $x = -\frac{\sqrt{6}}{2} \approx -1.22$  then turns back upwards to touch the  $x$ -axis at  $x = -1$ . This tells us that, despite what the calculator showed us the first time, there is a relative maximum occurring at  $x = -1$  and not a ‘flattened crossing’ as we originally

believed. After resizing the window, we see not only the relative maximum but also a relative minimum<sup>2</sup> just to the left of  $x = -1$  which shows us, once again, that Mathematics enhances the technology, instead of vice-versa.



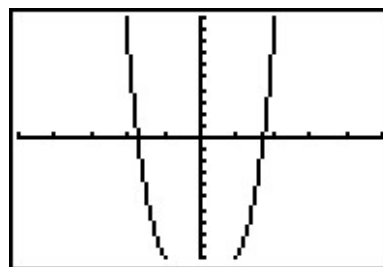
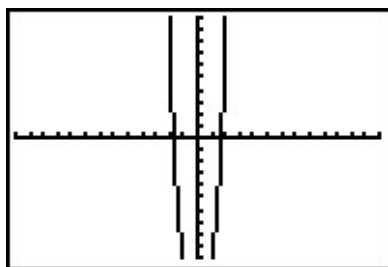
Our next example shows how even a mild-mannered polynomial can cause problems.

**Example 3.3.4.** Let  $f(x) = x^4 + x^2 - 12$ .

1. Use Cauchy's Bound to determine an interval in which all of the real zeros of  $f$  lie.
2. Use the Rational Zeros Theorem to determine a list of possible rational zeros of  $f$ .
3. Graph  $y = f(x)$  using your graphing calculator.
4. Find all of the real zeros of  $f$  and their multiplicities.

**Solution.**

1. Applying Cauchy's Bound, we find  $M = 12$ , so all of the real zeros lie in the interval  $[-13, 13]$ .
2. Applying the Rational Zeros Theorem with constant term  $a_0 = -12$  and leading coefficient  $a_4 = 1$ , we get the list  $\{\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12\}$ .
3. Graphing  $y = f(x)$  on the interval  $[-13, 13]$  produces the graph below on the left. Zooming in a bit gives the graph below on the right. Based on the graph, none of our rational zeros will work. (Do you see why not?)



<sup>2</sup>This is an example of what is called 'hidden behavior.'

4. From the graph, we know  $f$  has two real zeros, one positive, and one negative. Our only hope at this point is to try and find the zeros of  $f$  by setting  $f(x) = x^4 + x^2 - 12 = 0$  and solving. If we stare at this equation long enough, we may recognize it as a ‘quadratic in disguise’ or ‘quadratic in form’. In other words, we have three terms:  $x^4$ ,  $x^2$  and 12, and the exponent on the first term,  $x^4$ , is exactly twice that of the second term,  $x^2$ . We may rewrite this as  $(x^2)^2 + (x^2) - 12 = 0$ . To better see the forest for the trees, we momentarily replace  $x^2$  with the variable  $u$ . In terms of  $u$ , our equation becomes  $u^2 + u - 12 = 0$ , which we can readily factor as  $(u + 4)(u - 3) = 0$ . In terms of  $x$ , this means  $x^4 + x^2 - 12 = (x^2 - 3)(x^2 + 4) = 0$ . We get  $x^2 = 3$ , which gives us  $x = \pm\sqrt{3}$ , or  $x^2 = -4$ , which admits no real solutions. Since  $\sqrt{3} \approx 1.73$ , the two zeros match what we expected from the graph. In terms of multiplicity, the Factor Theorem guarantees  $(x - \sqrt{3})$  and  $(x + \sqrt{3})$  are factors of  $f(x)$ . Since  $f(x)$  can be factored as  $f(x) = (x^2 - 3)(x^2 + 4)$ , and  $x^2 + 4$  has no real zeros, the quantities  $(x - \sqrt{3})$  and  $(x + \sqrt{3})$  must both be factors of  $x^2 - 3$ . According to Theorem 3.7,  $x^2 - 3$  can have at most 2 zeros, counting multiplicity, hence each of  $\pm\sqrt{3}$  is a zero of  $f$  of multiplicity 1.  $\square$

The technique used to factor  $f(x)$  in Example 3.3.4 is called  **$u$ -substitution**. We shall see more of this technique in Section 5.3. In general, substitution can help us identify a ‘quadratic in disguise’ provided that there are exactly three terms and the exponent of the first term is exactly twice that of the second. It is entirely possible that a polynomial has no real roots at all, or worse, it has real roots but none of the techniques discussed in this section can help us find them exactly. In the latter case, we are forced to approximate, which in this subsection means we use the ‘Zero’ command on the graphing calculator.

### 3.3.2 FOR THOSE WISHING NOT TO USE A GRAPHING CALCULATOR

Suppose we wish to find the zeros of  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$  *without* using the calculator. In this subsection, we present some more advanced mathematical tools (theorems) to help us. Our first result is due to René Descartes.

**Theorem 3.10. Descartes’ Rule of Signs:** Suppose  $f(x)$  is the formula for a polynomial function written with descending powers of  $x$ .

- If  $P$  denotes the number of variations of sign in the formula for  $f(x)$ , then the number of positive real zeros (counting multiplicity) is one of the numbers  $\{P, P - 2, P - 4, \dots\}$ .
- If  $N$  denotes the number of variations of sign in the formula for  $f(-x)$ , then the number of negative real zeros (counting multiplicity) is one of the numbers  $\{N, N - 2, N - 4, \dots\}$ .

A few remarks are in order. First, to use Descartes’ Rule of Signs, we need to understand what is meant by a ‘**variation in sign**’ of a polynomial function. Consider  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ . If we focus on only the *signs* of the coefficients, we start with a (+), followed by another (+), then switch to (−), and stay (−) for the remaining two coefficients. Since the signs of the coefficients switched *once* as we read from left to right, we say that  $f(x)$  has *one* variation in sign. When

we speak of the variations in sign of a polynomial function  $f$  we assume the formula for  $f(x)$  is written with descending powers of  $x$ , as in Definition 3.1, and concern ourselves only with the nonzero coefficients. Second, unlike the Rational Zeros Theorem, Descartes' Rule of Signs gives us an estimate to the *number* of positive and negative real zeros, not the actual *value* of the zeros. Lastly, Descartes' Rule of Signs counts multiplicities. This means that, for example, if one of the zeros has multiplicity 2, Descartes' Rule of Signs would count this as *two* zeros. Lastly, note that the number of positive or negative real zeros always starts with the number of sign changes and decreases by an even number. For example, if  $f(x)$  has 7 sign changes, then, counting multiplicities,  $f$  has either 7, 5, 3 or 1 positive real zero. This implies that the graph of  $y = f(x)$  crosses the positive  $x$ -axis at least once. If  $f(-x)$  results in 4 sign changes, then, counting multiplicities,  $f$  has 4, 2 or 0 negative real zeros; hence, the graph of  $y = f(x)$  may not cross the negative  $x$ -axis at all. The proof of Descartes' Rule of Signs is a bit technical, and can be found [here](#).

**Example 3.3.5.** Let  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ . Use Descartes' Rule of Signs to determine the possible number and location of the real zeros of  $f$ .

**Solution.** As noted above, the variations in sign of  $f(x)$  is 1. This means, counting multiplicities,  $f$  has exactly 1 positive real zero. Since  $f(-x) = 2(-x)^4 + 4(-x)^3 - (-x)^2 - 6(-x) - 3 = 2x^4 - 4x^3 - x^2 + 6x - 3$  has 3 variations in sign,  $f$  has either 3 negative real zeros or 1 negative real zero, counting multiplicities.  $\square$

Cauchy's Bound gives us a general bound on the zeros of a polynomial function. Our next result helps us determine bounds on the real zeros of a polynomial as we synthetically divide which are often sharper<sup>3</sup> bounds than Cauchy's Bound.

**Theorem 3.11. Upper and Lower Bounds:** Suppose  $f$  is a polynomial of degree  $n \geq 1$ .

- If  $c > 0$  is synthetically divided into  $f$  and all of the numbers in the final line of the division tableau have the same signs, then  $c$  is an upper bound for the real zeros of  $f$ . That is, there are no real zeros greater than  $c$ .
- If  $c < 0$  is synthetically divided into  $f$  and the numbers in the final line of the division tableau alternate signs, then  $c$  is a lower bound for the real zeros of  $f$ . That is, there are no real zeros less than  $c$ .

**NOTE:** If the number 0 occurs in the final line of the division tableau in either of the above cases, it can be treated as (+) or (-) as needed.

The Upper and Lower Bounds Theorem works because of Theorem 3.4. For the upper bound part of the theorem, suppose  $c > 0$  is divided into  $f$  and the resulting line in the division tableau contains, for example, all nonnegative numbers. This means  $f(x) = (x - c)q(x) + r$ , where the coefficients of the quotient polynomial and the remainder are nonnegative. (Note that the leading coefficient of  $q$  is the same as  $f$  so  $q(x)$  is not the zero polynomial.) If  $b > c$ , then  $f(b) = (b - c)q(b) + r$ , where  $(b - c)$  and  $q(b)$  are both positive and  $r \geq 0$ . Hence  $f(b) > 0$  which shows  $b$  cannot be a zero of  $f$ . Thus no real number  $b > c$  can be a zero of  $f$ , as required. A similar argument proves

<sup>3</sup>That is, better, or more accurate.

$f(b) < 0$  if all of the numbers in the final line of the synthetic division tableau are non-positive. To prove the lower bound part of the theorem, we note that a lower bound for the negative real zeros of  $f(x)$  is an upper bound for the positive real zeros of  $f(-x)$ . Applying the upper bound portion to  $f(-x)$  gives the result. (Do you see where the alternating signs come in?) With the additional mathematical machinery of Descartes' Rule of Signs and the Upper and Lower Bounds Theorem, we can find the real zeros of  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$  without the use of a graphing calculator.

**Example 3.3.6.** Let  $f(x) = 2x^4 + 4x^3 - x^2 - 6x - 3$ .

1. Find all of the real zeros of  $f$  and their multiplicities.
2. Sketch the graph of  $y = f(x)$ .

**Solution.**

1. We know from Cauchy's Bound that all of the real zeros lie in the interval  $[-4, 4]$  and that our possible rational zeros are  $\pm \frac{1}{2}$ ,  $\pm 1$ ,  $\pm \frac{3}{2}$  and  $\pm 3$ . Descartes' Rule of Signs guarantees us at least one negative real zero and exactly one positive real zero, counting multiplicity. We try our positive rational zeros, starting with the smallest,  $\frac{1}{2}$ . Since the remainder isn't zero, we know  $\frac{1}{2}$  isn't a zero. Sadly, the final line in the division tableau has both positive and negative numbers, so  $\frac{1}{2}$  is not an upper bound. The only information we get from this division is courtesy of the Remainder Theorem which tells us  $f(\frac{1}{2}) = -\frac{45}{8}$  so the point  $(\frac{1}{2}, -\frac{45}{8})$  is on the graph of  $f$ . We continue to our next possible zero, 1. As before, the only information we can glean from this is that  $(1, -4)$  is on the graph of  $f$ . When we try our next possible zero,  $\frac{3}{2}$ , we get that it is not a zero, and we also see that it is an upper bound on the zeros of  $f$ , since all of the numbers in the final line of the division tableau are positive. This means there is no point trying our last possible rational zero, 3. Descartes' Rule of Signs guaranteed us a positive real zero, and at this point we have shown this zero is irrational. Furthermore, the Intermediate Value Theorem, Theorem 3.1, tells us the zero lies between 1 and  $\frac{3}{2}$ , since  $f(1) < 0$  and  $f(\frac{3}{2}) > 0$ .

$$\begin{array}{r|rrrrr} \frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 1 & \frac{5}{2} & \frac{3}{4} & -\frac{21}{8} \\ \hline & 2 & 5 & \frac{3}{2} & -\frac{21}{4} & \boxed{-\frac{45}{8}} \end{array} \quad
 \begin{array}{r|rrrrr} 1 & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 2 & 6 & 5 & -1 \\ \hline & 2 & 6 & 5 & -1 & \boxed{-4} \end{array} \quad
 \begin{array}{r|rrrrr} \frac{3}{2} & 2 & 4 & -1 & -6 & -3 \\ & \downarrow & 3 & \frac{21}{2} & \frac{57}{4} & \frac{99}{8} \\ \hline & 2 & 7 & \frac{19}{2} & \frac{33}{4} & \boxed{\frac{75}{8}} \end{array}$$

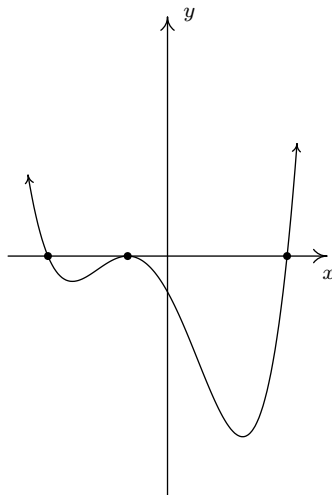
We now turn our attention to negative real zeros. We try the largest possible zero,  $-\frac{1}{2}$ . Synthetic division shows us it is not a zero, nor is it a lower bound (since the numbers in the final line of the division tableau do not alternate), so we proceed to  $-1$ . This division shows  $-1$  is a zero. Descartes' Rule of Signs told us that we may have up to three negative real zeros, counting multiplicity, so we try  $-1$  again, and it works once more. At this point, we have taken  $f$ , a fourth degree polynomial, and performed two successful divisions. Our quotient polynomial is quadratic, so we look at it to find the remaining zeros.

$$\begin{array}{r|rrrrr}
 -\frac{1}{2} & 2 & 4 & -1 & -6 & -3 \\
 & \downarrow & -1 & -\frac{3}{2} & \frac{5}{4} & \frac{19}{8} \\
 \hline
 & 2 & 3 & -\frac{5}{2} & -\frac{19}{4} & \boxed{-\frac{5}{8}}
 \end{array}$$

$$\begin{array}{r|rrrrr}
 -1 & 2 & 4 & -1 & -6 & -3 \\
 & \downarrow & -2 & -2 & 3 & 3 \\
 -1 & \hline
 & 2 & 2 & -3 & -3 & \boxed{0} \\
 & \downarrow & -2 & 0 & 3 & \\
 & \hline
 & 2 & 0 & -3 & \boxed{0}
 \end{array}$$

Setting the quotient polynomial equal to zero yields  $2x^2 - 3 = 0$ , so that  $x^2 = \frac{3}{2}$ , or  $x = \pm \frac{\sqrt{6}}{2}$ . Descartes' Rule of Signs tells us that the positive real zero we found,  $\frac{\sqrt{6}}{2}$ , has multiplicity 1. Descartes also tells us the total multiplicity of negative real zeros is 3, which forces  $-1$  to be a zero of multiplicity 2 and  $-\frac{\sqrt{6}}{2}$  to have multiplicity 1.

2. We know the end behavior of  $y = f(x)$  resembles that of its leading term  $y = 2x^4$ . This means that the graph enters the scene in Quadrant II and exits in Quadrant I. Since  $\pm \frac{\sqrt{6}}{2}$  are zeros of odd multiplicity, we have that the graph crosses through the  $x$ -axis at the points  $(-\frac{\sqrt{6}}{2}, 0)$  and  $(\frac{\sqrt{6}}{2}, 0)$ . Since  $-1$  is a zero of multiplicity 2, the graph of  $y = f(x)$  touches and rebounds off the  $x$ -axis at  $(-1, 0)$ . Putting this together, we get



□

You can see why the ‘no calculator’ approach is not very popular these days. It requires more computation and more theorems than the alternative.<sup>4</sup> In general, no matter how many theorems you throw at a polynomial, it may well be impossible<sup>5</sup> to find their zeros exactly. The polynomial  $f(x) = x^5 - x - 1$  is one such beast.<sup>6</sup> According to Descartes' Rule of Signs,  $f$  has exactly one positive real zero, and it could have two negative real zeros, or none at all. The Rational Zeros

<sup>4</sup>This is apparently a bad thing.

<sup>5</sup>We don't use this word lightly; it can be proven that the zeros of some polynomials cannot be expressed using the usual algebraic symbols.

<sup>6</sup>See this [page](#).



Test gives us  $\pm 1$  as rational zeros to try but neither of these work since  $f(1) = f(-1) = -1$ . If we try the substitution technique we used in Example 3.3.4, we find  $f(x)$  has three terms, but the exponent on the  $x^5$  isn't exactly twice the exponent on  $x$ . How could we go about approximating the positive zero without resorting to the 'Zero' command of a graphing calculator? We use the **Bisection Method**. The first step in the Bisection Method is to find an interval on which  $f$  changes sign. We know  $f(1) = -1$  and we find  $f(2) = 29$ . By the Intermediate Value Theorem, we know that the zero of  $f$  lies in the interval  $[1, 2]$ . Next, we 'bisect' this interval and find the midpoint is 1.5. We have that  $f(1.5) \approx 5.09$ . This means that our zero is between 1 and 1.5, since  $f$  changes sign on this interval. Now, we 'bisect' the interval  $[1, 1.5]$  and find  $f(1.25) \approx 0.80$ , so now we have the zero between 1 and 1.25. Bisecting  $[1, 1.25]$ , we find  $f(1.125) \approx -0.32$ , which means the zero of  $f$  is between 1.125 and 1.25. We continue in this fashion until we have 'sandwiched' the zero between two numbers which differ by no more than a desired accuracy. You can think of the Bisection Method as reversing the sign diagram process: instead of finding the zeros and checking the sign of  $f$  using test values, we are using test values to determine where the signs switch to find the zeros. It is a slow and tedious, yet fool-proof, method for approximating a real zero.

Our next example reminds us of the role finding zeros plays in solving equations and inequalities.

**Example 3.3.7.**

1. Find all of the real solutions to the equation  $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$ .
2. Solve the inequality  $2x^5 + 6x^3 + 3 \leq 3x^4 + 8x^2$ .
3. Interpret your answer to part 2 graphically, and verify using a graphing calculator.

**Solution.**

1. Finding the real solutions to  $2x^5 + 6x^3 + 3 = 3x^4 + 8x^2$  is the same as finding the real solutions to  $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 = 0$ . In other words, we are looking for the real zeros of  $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$ . Using the techniques developed in this section, we get

$$\begin{array}{r|rrrrrr}
 1 & 2 & -3 & 6 & -8 & 0 & 3 \\
 & \downarrow & 2 & -1 & 5 & -3 & -3 \\
 1 & 2 & -1 & 5 & -3 & -3 & \boxed{0} \\
 & \downarrow & 2 & 1 & 6 & 3 & \\
 -\frac{1}{2} & 2 & 1 & 6 & 3 & \boxed{0} & \\
 & \downarrow & -1 & 0 & -3 & & \\
 & 2 & 0 & 6 & \boxed{0} & & 
 \end{array}$$

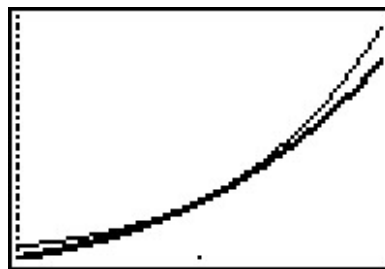
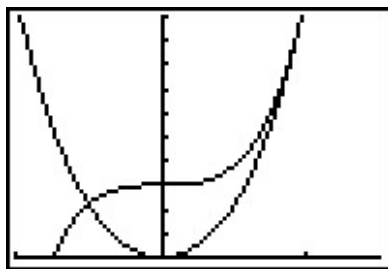
The quotient polynomial is  $2x^2 + 6$  which has no real zeros so we get  $x = -\frac{1}{2}$  and  $x = 1$ .

2. To solve this nonlinear inequality, we follow the same guidelines set forth in Section 2.4: we get 0 on one side of the inequality and construct a sign diagram. Our original inequality can be rewritten as  $2x^5 - 3x^4 + 6x^3 - 8x^2 + 3 \leq 0$ . We found the zeros of  $p(x) = 2x^5 - 3x^4 + 6x^3 - 8x^2 + 3$  in part 1 to be  $x = -\frac{1}{2}$  and  $x = 1$ . We construct our sign diagram as before.

$$\begin{array}{ccccccc}
 & (-) & 0 & (+) & 0 & (+) & \\
 & \leftarrow & & & & & \rightarrow \\
 & \uparrow & -\frac{1}{2} & \uparrow & \uparrow & & \\
 & -1 & & 0 & 1 & 2 & 
 \end{array}$$

The solution to  $p(x) < 0$  is  $(-\infty, -\frac{1}{2})$ , and we know  $p(x) = 0$  at  $x = -\frac{1}{2}$  and  $x = 1$ . Hence, the solution to  $p(x) \leq 0$  is  $(-\infty, -\frac{1}{2}] \cup \{1\}$ .

3. To interpret this solution graphically, we set  $f(x) = 2x^5 + 6x^3 + 3$  and  $g(x) = 3x^4 + 8x^2$ . We recall that the solution to  $f(x) \leq g(x)$  is the set of  $x$  values for which the graph of  $f$  is below the graph of  $g$  (where  $f(x) < g(x)$ ) along with the  $x$  values where the two graphs intersect ( $f(x) = g(x)$ ). Graphing  $f$  and  $g$  on the calculator produces the picture on the lower left. (The end behavior should tell you which is which.) We see that the graph of  $f$  is below the graph of  $g$  on  $(-\infty, -\frac{1}{2})$ . However, it is difficult to see what is happening near  $x = 1$ . Zooming in (and making the graph of  $g$  thicker), we see that the graphs of  $f$  and  $g$  do intersect at  $x = 1$ , but the graph of  $g$  remains below the graph of  $f$  on either side of  $x = 1$ .

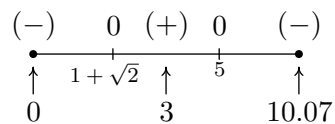


□

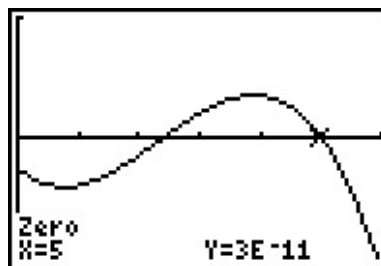
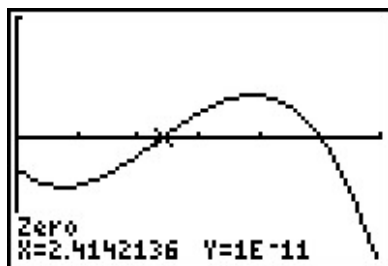
Our last example revisits an application from page 247 in the Exercises of Section 3.1.

**Example 3.3.8.** Suppose the profit  $P$ , in *thousands* of dollars, from producing and selling  $x$  hundred LCD TVs is given by  $P(x) = -5x^3 + 35x^2 - 45x - 25$ ,  $0 \leq x \leq 10.07$ . How many TVs should be produced to make a profit? Check your answer using a graphing utility.

**Solution.** To ‘make a profit’ means to solve  $P(x) = -5x^3 + 35x^2 - 45x - 25 > 0$ , which we do analytically using a sign diagram. To simplify things, we first factor out the  $-5$  common to all the coefficients to get  $-5(x^3 - 7x^2 + 9x - 5) > 0$ , so we can just focus on finding the zeros of  $f(x) = x^3 - 7x^2 + 9x + 5$ . The possible rational zeros of  $f$  are  $\pm 1$  and  $\pm 5$ , and going through the usual computations, we find  $x = 5$  is the only rational zero. Using this, we factor  $f(x) = x^3 - 7x^2 + 9x + 5 = (x - 5)(x^2 - 2x - 1)$ , and we find the remaining zeros by applying the Quadratic Formula to  $x^2 - 2x - 1 = 0$ . We find three real zeros,  $x = 1 - \sqrt{2} = -0.414\dots$ ,  $x = 1 + \sqrt{2} = 2.414\dots$ , and  $x = 5$ , of which only the last two fall in the applied domain of  $[0, 10.07]$ . We choose  $x = 0$ ,  $x = 3$  and  $x = 10.07$  as our test values and plug them into the function  $P(x) = -5x^3 + 35x^2 - 45x - 25$  (not  $f(x) = x^3 - 7x^2 + 9x - 5$ ) to get the sign diagram below.



We see immediately that  $P(x) > 0$  on  $(1 + \sqrt{2}, 5)$ . Since  $x$  measures the number of TVs in *hundreds*,  $x = 1 + \sqrt{2}$  corresponds to 241.4... TVs. Since we can't produce a fractional part of a TV, we need to choose between producing 241 and 242 TVs. From the sign diagram, we see that  $P(2.41) < 0$  but  $P(2.42) > 0$  so, in this case we take the next *larger* integer value and set the minimum production to 242 TVs. At the other end of the interval, we have  $x = 5$  which corresponds to 500 TVs. Here, we take the next *smaller* integer value, 499 TVs to ensure that we make a profit. Hence, in order to make a profit, at least 242, but no more than 499 TVs need to be produced. To check our answer using a calculator, we graph  $y = P(x)$  and make use of the 'Zero' command. We see that the calculator approximations bear out our analysis.<sup>7</sup>



□

<sup>7</sup>Note that the  $y$ -coordinates of the points here aren't registered as 0. They are expressed in Scientific Notation. For instance,  $1E - 11$  corresponds to 0.00000000001, which is pretty close in the calculator's eyes<sup>8</sup> to 0.

<sup>8</sup>but not a Mathematician's

## 3.3.3 EXERCISES

In Exercises 1 - 10, for the given polynomial:

- Use Cauchy's Bound to find an interval containing all of the real zeros.
- Use the Rational Zeros Theorem to make a list of possible rational zeros.
- Use Descartes' Rule of Signs to list the possible number of positive and negative real zeros, counting multiplicities.

1.  $f(x) = x^3 - 2x^2 - 5x + 6$

2.  $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$

3.  $f(x) = x^4 - 9x^2 - 4x + 12$

4.  $f(x) = x^3 + 4x^2 - 11x + 6$

5.  $f(x) = x^3 - 7x^2 + x - 7$

6.  $f(x) = -2x^3 + 19x^2 - 49x + 20$

7.  $f(x) = -17x^3 + 5x^2 + 34x - 10$

8.  $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

9.  $f(x) = 3x^3 + 3x^2 - 11x - 10$

10.  $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

In Exercises 11 - 30, find the real zeros of the polynomial using the techniques specified by your instructor. State the multiplicity of each real zero.

11.  $f(x) = x^3 - 2x^2 - 5x + 6$

12.  $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$

13.  $f(x) = x^4 - 9x^2 - 4x + 12$

14.  $f(x) = x^3 + 4x^2 - 11x + 6$

15.  $f(x) = x^3 - 7x^2 + x - 7$

16.  $f(x) = -2x^3 + 19x^2 - 49x + 20$

17.  $f(x) = -17x^3 + 5x^2 + 34x - 10$

18.  $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$

19.  $f(x) = 3x^3 + 3x^2 - 11x - 10$

20.  $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$

21.  $f(x) = 9x^3 - 5x^2 - x$

22.  $f(x) = 6x^4 - 5x^3 - 9x^2$

23.  $f(x) = x^4 + 2x^2 - 15$

24.  $f(x) = x^4 - 9x^2 + 14$

25.  $f(x) = 3x^4 - 14x^2 - 5$

26.  $f(x) = 2x^4 - 7x^2 + 6$

27.  $f(x) = x^6 - 3x^3 - 10$

28.  $f(x) = 2x^6 - 9x^3 + 10$

29.  $f(x) = x^5 - 2x^4 - 4x + 8$

30.  $f(x) = 2x^5 + 3x^4 - 18x - 27$

In Exercises 31 - 33, use your calculator,<sup>9</sup> to help you find the real zeros of the polynomial. State the multiplicity of each real zero.

31.  $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$

32.  $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$

33.  $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$

34. Find the real zeros of  $f(x) = x^3 - \frac{1}{12}x^2 - \frac{7}{72}x + \frac{1}{72}$  by first finding a polynomial  $q(x)$  with integer coefficients such that  $q(x) = N \cdot f(x)$  for some integer  $N$ . (Recall that the Rational Zeros Theorem required the polynomial in question to have integer coefficients.) Show that  $f$  and  $q$  have the same real zeros.

In Exercises 35 - 44, solve the polynomial inequality and state your answer using interval notation.

35.  $-2x^3 + 19x^2 - 49x + 20 > 0$

36.  $x^4 - 9x^2 \leq 4x - 12$

37.  $(x - 1)^2 \geq 4$

38.  $4x^3 \geq 3x + 1$

39.  $x^4 \leq 16 + 4x - x^3$

40.  $3x^2 + 2x < x^4$

41.  $\frac{x^3 + 2x^2}{2} < x + 2$

42.  $\frac{x^3 + 20x}{8} \geq x^2 + 2$

43.  $2x^4 > 5x^2 + 3$

44.  $x^6 + x^3 \geq 6$

45. In Example 3.1.3 in Section 3.1, a box with no top is constructed from a 10 inch  $\times$  12 inch piece of cardboard by cutting out congruent squares from each corner of the cardboard and then folding the resulting tabs. We determined the volume of that box (in cubic inches) is given by  $V(x) = 4x^3 - 44x^2 + 120x$ , where  $x$  denotes the length of the side of the square which is removed from each corner (in inches),  $0 < x < 5$ . Solve the inequality  $V(x) \geq 80$  analytically and interpret your answer in the context of that example.

46. From Exercise 32 in Section 3.1,  $C(x) = .03x^3 - 4.5x^2 + 225x + 250$ , for  $x \geq 0$  models the cost, in dollars, to produce  $x$  PortaBoy game systems. If the production budget is \$5000, find the number of game systems which can be produced and still remain under budget.

47. Let  $f(x) = 5x^7 - 33x^6 + 3x^5 - 71x^4 - 597x^3 + 2097x^2 - 1971x + 567$ . With the help of your classmates, find the  $x$ - and  $y$ - intercepts of the graph of  $f$ . Find the intervals on which the function is increasing, the intervals on which it is decreasing and the local extrema. Sketch the graph of  $f$ , using more than one picture if necessary to show all of the important features of the graph.

48. With the help of your classmates, create a list of five polynomials with different degrees whose real zeros cannot be found using any of the techniques in this section.

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<sup>9</sup>You *can* do these without your calculator, but it may test your mettle!

## 3.3.4 ANSWERS

1. For  $f(x) = x^3 - 2x^2 - 5x + 6$ 
  - All of the real zeros lie in the interval  $[-7, 7]$
  - Possible rational zeros are  $\pm 1, \pm 2, \pm 3$
  - There are 2 or 0 positive real zeros; there is 1 negative real zero
2. For  $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$ 
  - All of the real zeros lie in the interval  $[-33, 33]$
  - Possible rational zeros are  $\pm 1, \pm 2, \pm 4, \pm 8, \pm 16, \pm 32$
  - There is 1 positive real zero; there are 3 or 1 negative real zeros
3. For  $f(x) = x^4 - 9x^2 - 4x + 12$ 
  - All of the real zeros lie in the interval  $[-13, 13]$
  - Possible rational zeros are  $\pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 12$
  - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
4. For  $f(x) = x^3 + 4x^2 - 11x + 6$ 
  - All of the real zeros lie in the interval  $[-7, 7]$
  - Possible rational zeros are  $\pm 1, \pm 2, \pm 3, \pm 6$
  - There are 2 or 0 positive real zeros; there is 1 negative real zero
5. For  $f(x) = x^3 - 7x^2 + x - 7$ 
  - All of the real zeros lie in the interval  $[-8, 8]$
  - Possible rational zeros are  $\pm 1, \pm 7$
  - There are 3 or 1 positive real zeros; there are no negative real zeros
6. For  $f(x) = -2x^3 + 19x^2 - 49x + 20$ 
  - All of the real zeros lie in the interval  $[-\frac{51}{2}, \frac{51}{2}]$
  - Possible rational zeros are  $\pm\frac{1}{2}, \pm 1, \pm 2, \pm\frac{5}{2}, \pm 4, \pm 5, \pm 10, \pm 20$
  - There are 3 or 1 positive real zeros; there are no negative real zeros
7. For  $f(x) = -17x^3 + 5x^2 + 34x - 10$ 
  - All of the real zeros lie in the interval  $[-3, 3]$
  - Possible rational zeros are  $\pm\frac{1}{17}, \pm\frac{2}{17}, \pm\frac{5}{17}, \pm\frac{10}{17}, \pm 1, \pm 2, \pm 5, \pm 10$
  - There are 2 or 0 positive real zeros; there is 1 negative real zero

8. For  $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$
- All of the real zeros lie in the interval  $[-\frac{4}{3}, \frac{4}{3}]$
  - Possible rational zeros are  $\pm\frac{1}{36}, \pm\frac{1}{18}, \pm\frac{1}{12}, \pm\frac{1}{9}, \pm\frac{1}{6}, \pm\frac{1}{4}, \pm\frac{1}{3}, \pm\frac{1}{2}, \pm 1$
  - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
9. For  $f(x) = 3x^3 + 3x^2 - 11x - 10$
- All of the real zeros lie in the interval  $[-\frac{14}{3}, \frac{14}{3}]$
  - Possible rational zeros are  $\pm\frac{1}{3}, \pm\frac{2}{3}, \pm\frac{5}{3}, \pm\frac{10}{3}, \pm 1, \pm 2, \pm 5, \pm 10$
  - There is 1 positive real zero; there are 2 or 0 negative real zeros
10. For  $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$
- All of the real zeros lie in the interval  $[-\frac{9}{2}, \frac{9}{2}]$
  - Possible rational zeros are  $\pm\frac{1}{2}, \pm 1, \pm\frac{3}{2}, \pm 3$
  - There are 2 or 0 positive real zeros; there are 2 or 0 negative real zeros
11.  $f(x) = x^3 - 2x^2 - 5x + 6$   
 $x = -2, x = 1, x = 3$  (each has mult. 1)
12.  $f(x) = x^4 + 2x^3 - 12x^2 - 40x - 32$   
 $x = -2$  (mult. 3),  $x = 4$  (mult. 1)
13.  $f(x) = x^4 - 9x^2 - 4x + 12$   
 $x = -2$  (mult. 2),  $x = 1$  (mult. 1),  $x = 3$  (mult. 1)
14.  $f(x) = x^3 + 4x^2 - 11x + 6$   
 $x = -6$  (mult. 1),  $x = 1$  (mult. 2)
15.  $f(x) = x^3 - 7x^2 + x - 7$   
 $x = 7$  (mult. 1)
16.  $f(x) = -2x^3 + 19x^2 - 49x + 20$   
 $x = \frac{1}{2}, x = 4, x = 5$  (each has mult. 1)
17.  $f(x) = -17x^3 + 5x^2 + 34x - 10$   
 $x = \frac{5}{17}, x = \pm\sqrt{2}$  (each has mult. 1)
18.  $f(x) = 36x^4 - 12x^3 - 11x^2 + 2x + 1$   
 $x = \frac{1}{2}$  (mult. 2),  $x = -\frac{1}{3}$  (mult. 2)
19.  $f(x) = 3x^3 + 3x^2 - 11x - 10$   
 $x = -2, x = \frac{3 \pm \sqrt{69}}{6}$  (each has mult. 1)

20.  $f(x) = 2x^4 + x^3 - 7x^2 - 3x + 3$   
 $x = -1, x = \frac{1}{2}, x = \pm\sqrt{3}$  (each mult. 1)
21.  $f(x) = 9x^3 - 5x^2 - x$   
 $x = 0, x = \frac{5 \pm \sqrt{61}}{18}$  (each has mult. 1)
22.  $f(x) = 6x^4 - 5x^3 - 9x^2$   
 $x = 0$  (mult. 2),  $x = \frac{5 \pm \sqrt{241}}{12}$  (each has mult. 1)
23.  $f(x) = x^4 + 2x^2 - 15$   
 $x = \pm\sqrt{3}$  (each has mult. 1)
24.  $f(x) = x^4 - 9x^2 + 14$   
 $x = \pm\sqrt{2}, x = \pm\sqrt{7}$  (each has mult. 1)
25.  $f(x) = 3x^4 - 14x^2 - 5$   
 $x = \pm\sqrt{5}$  (each has mult. 1)
26.  $f(x) = 2x^4 - 7x^2 + 6$   
 $x = \pm\frac{\sqrt{6}}{2}, x = \pm\sqrt{2}$  (each has mult. 1)
27.  $f(x) = x^6 - 3x^3 - 10$   
 $x = \sqrt[3]{-2} = -\sqrt[3]{2}, x = \sqrt[3]{5}$  (each has mult. 1)
28.  $f(x) = 2x^6 - 9x^3 + 10$   
 $x = \frac{\sqrt[3]{20}}{2}, x = \sqrt[3]{2}$  (each has mult. 1)
29.  $f(x) = x^5 - 2x^4 - 4x + 8$   
 $x = 2, x = \pm\sqrt{2}$  (each has mult. 1)
30.  $f(x) = 2x^5 + 3x^4 - 18x - 27$   
 $x = -\frac{3}{2}, x = \pm\sqrt{3}$  (each has mult. 1)
31.  $f(x) = x^5 - 60x^3 - 80x^2 + 960x + 2304$   
 $x = -4$  (mult. 3),  $x = 6$  (mult. 2)
32.  $f(x) = 25x^5 - 105x^4 + 174x^3 - 142x^2 + 57x - 9$   
 $x = \frac{3}{5}$  (mult. 2),  $x = 1$  (mult. 3)
33.  $f(x) = 90x^4 - 399x^3 + 622x^2 - 399x + 90$   
 $x = \frac{2}{3}, x = \frac{3}{2}, x = \frac{5}{3}, x = \frac{3}{5}$  (each has mult. 1)
34. We choose  $q(x) = 72x^3 - 6x^2 - 7x + 1 = 72 \cdot f(x)$ . Clearly  $f(x) = 0$  if and only if  $q(x) = 0$  so they have the same real zeros. In this case,  $x = -\frac{1}{3}, x = \frac{1}{6}$  and  $x = \frac{1}{4}$  are the real zeros of both  $f$  and  $q$ .



35.  $(-\infty, \frac{1}{2}) \cup (4, 5)$

36.  $\{-2\} \cup [1, 3]$

37.  $(-\infty, -1] \cup [3, \infty)$

38.  $\left\{-\frac{1}{2}\right\} \cup [1, \infty)$

39.  $[-2, 2]$

40.  $(-\infty, -1) \cup (-1, 0) \cup (2, \infty)$

41.  $(-\infty, -2) \cup (-\sqrt{2}, \sqrt{2})$

42.  $\{2\} \cup [4, \infty)$

43.  $(-\infty, -\sqrt{3}) \cup (\sqrt{3}, \infty)$

44.  $(-\infty, -\sqrt[3]{3}) \cup (\sqrt[3]{2}, \infty)$

45.  $V(x) \geq 80$  on  $[1, 5 - \sqrt{5}] \cup [5 + \sqrt{5}, \infty)$ . Only the portion  $[1, 5 - \sqrt{5}]$  lies in the applied domain, however. In the context of the problem, this says for the volume of the box to be at least 80 cubic inches, the square removed from each corner needs to have a side length of at least 1 inch, but no more than  $5 - \sqrt{5} \approx 2.76$  inches.

46.  $C(x) \leq 5000$  on (approximately)  $(-\infty, 82.18]$ . The portion of this which lies in the applied domain is  $(0, 82.18]$ . Since  $x$  represents the number of game systems, we check  $C(82) = 4983.04$  and  $C(83) = 5078.11$ , so to remain within the production budget, anywhere between 1 and 82 game systems can be produced.